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# Fuzzy $h$-ideals extension in $\Gamma$-hemirings

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Abstract. In this paper the concept of the extension of fuzzy $h$-ideals in $\Gamma$-hemiring is introduced and some of its properties are investigated. Specially we have studied the extension of prime fuzzy $h$-ideals in $\Gamma$-hemirings and gave its characterization.

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## 1. Introduction

Recently, there have been appearing several papers on fuzzy algebraic structure. In algebra, semirings(hemirings) appear in a natural way in theory of automata, formal languages, theoretical computer sciences etc.. These areas and fuzzy logic have useful application in control engineering. This may be a reason why there have been attempts to fuzzify basic concepts of semirings (hemirings) theory. The concept of fuzzy set was introduced by Zadeh [14]. Jun and Lee [5] applied the concept of fuzzy sets to the theory of $\Gamma$-rings. The notion of $\Gamma$-semiring was introduced by Rao [8] as a generalization of $\Gamma$-ring as well as of semiring. $\Gamma$-semirings also include ternary semirings and provide algebraic home to non-positives cones of totally ordered rings. Henriksen [2], Iizuka [3] and LaTorre [6] investigated $h$-ideals and k-ideals in hemirings to amend the gap between ring ideals and semiring ideals. These concepts are extended to $\Gamma$-semiring by Rao [8], Dutta and Sardar [1]. Jun et al. [4] and Zhan et al. [15] studied these ideals in hemirings in terms of fuzzy subsets. We ([10, 11]) extended these concepts to the theory of $\Gamma$-hemirings. Motivated by Xie [13], as a continuation of this we study the concept of the extension of fuzzy $h$-ideals in $\Gamma$-hemirings and investigate some of its properties.

## 2. Preliminaries

**Definition 2.1.** Let $S$ and $\Gamma$ be two additive commutative semigroups with zero. Then $S$ is called a $\Gamma$-*hemiring* if there exists a mapping $S\times\Gamma\times S \to S$, $(a,\alpha,b)\mapsto a\alpha b$, satisfying the following conditions:

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a,b,c\in S$ and for all $\alpha,\beta\in\Gamma$. In addition, if $\Gamma$ is also $S$-hemiring, we call $S$ to be both-sided $\Gamma$-hemiring. For simplification we write 0 instead of $0_S$ and $0_\Gamma$. A $\Gamma$-hemiring $S$ is called *commutative* if $a\alpha b = b\alpha a$ for all $a,b\in S$ and $\alpha\in\Gamma$.

Throughout this paper $S$ denotes a both-sided $\Gamma$-hemiring with zero.

**Definition 2.2.** A left ideal $A$ of a $\Gamma$-hemiring $S$ is called a *left $h$-ideal* if for any $x,z\in S$ and $a,b\in A$, $x+a+z=b+z \Rightarrow x\in A$. A right $h$-ideal is defined analogously.

**Definition 2.3.** Let $S$ be a $\Gamma$-hemiring. A proper $h$-ideal $I$ of $S$ is said to be *prime* if for any two $h$-ideals $H$ and $K$ of $S$, $H\Gamma K\subseteq I$ implies that either $H\subseteq I$ or $K\subseteq I$.

**Theorem 2.4** ([11])**.** *If $I$ is an $h$-ideal of a $\Gamma$-hemiring $S$ then the following conditions are equivalent:*

(i) *$I$ is a prime $h$-ideal of $S$.*
(ii) *If $a\Gamma S\Gamma b\subseteq I$ then either $a\in I$ or $b\in I$ where $a,b\in S$.*

**Definition 2.5** ([12])**.** Let $\mu$ and $\theta$ be two fuzzy sets of a $\Gamma$-hemiring $S$. Define a generalized $h$-product of $\mu$ and $\theta$ by

$$\mu o_h\theta(x) = \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\mu(a_i),\mu(c_i),\theta(b_i),\theta(d_i)\}\}]$$
$$= 0, \text{if x cannot be expressed as above}$$

where $x,z,a_i,b_i,c_i,d_i\in S$ and $\gamma_i,\delta_i\in\Gamma$, for $i=1,2,\cdots,n$.

**Definition 2.6.** Let $\mu$ be the non empty fuzzy subset of a $\Gamma$-hemiring $S$ (i.e. $\mu(x)\neq 0$ for some $x\in S$). Then $\mu$ is called a *fuzzy left $h$-ideal* [resp. *fuzzy right $h$-ideal*] of $S$ if

(i) $\mu(x+y)\geq\min\{\mu(x),\mu(y)\}$,
(ii) $\mu(x\gamma y)\geq\mu(y)$ [resp. $\mu(x\gamma y)\geq\mu(x)$] for all $x,y\in S,\gamma\in\Gamma$,
(iii) For all $x,a,b,z\in S$, $x+a+z=b+z$ implies $\mu(x)\geq\min\{\mu(a),\mu(b)\}$.

A fuzzy ideal of a $\Gamma$-hemiring $S$ is a non empty fuzzy subset of $S$ which is a fuzzy left ideal as well as a fuzzy right ideal of $S$. A fuzzy $h$-ideal $\mu$ of $S$ is called a *fuzzy $h$-bi-ideal, fuzzy $h$-interior ideal* if for all $x,y,z,a,b\in S$ and $\alpha,\beta\in\Gamma$, $\mu(x\alpha y\beta z)\geq\min\{\mu(x),\mu(z)\}$, $\mu(x\alpha y\beta z)\geq\mu(y)$, respectively. A fuzzy subset $\mu$ of a $\Gamma$-hemiring

$S$ is called *fuzzy h-quasi-ideal* if $\mu$ satisfies (i) and (iii) along with the condition $(\mu o_h \chi_S) \cap (\chi_S o_h \mu) \subseteq \mu$, where $\chi_S$ is the characteristic function of $S$.

Now we recall following definitions and result from [7] for subsequent use.

**Definition 2.7.** Let $\mu$ and $\theta$ be two fuzzy sets of a $\Gamma$-hemiring $S$. Define $h$-product of $\mu$ and $\theta$ by

$$\begin{aligned} \mu\Gamma_h\theta(x) &= \sup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}[\min\{\mu(a_1),\mu(a_2),\theta(b_1),\theta(b_2)\}] \\ &= 0, \text{ if } x \text{ cannot be expressed as } x+a_1\gamma b_1+z=a_2\delta b_2+z \end{aligned}$$

for $x, z, a_1, a_2, b_1, b_2 \in S$ and $\gamma, \delta \in \Gamma$.

**Definition 2.8.** A fuzzy $h$-ideal $\mu$ of a $\Gamma$-hemiring $S$ is said to be *prime* (*semiprime*) if $\mu$ is not a constant function and for any two fuzzy $h$-ideals $\sigma$ and $\theta$ of S, $\sigma\Gamma_h\theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$ (resp. $\theta\Gamma_h\theta \subseteq \mu$ implies $\theta \subseteq \mu$).

**Theorem 2.9.** *Let $\mu$ be a fuzzy h-ideal of S. Then $\mu$ is a prime fuzzy h-ideal of S if and only if the following conditions hold*

(i) $\mu(0) = 1$,
(ii) $Im\ \mu = \{1, \alpha\}, \quad \alpha \in [0, 1)$,
(iii) $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ *is a prime h-ideal of S.*

## 3. Fuzzy $h$-ideal extension in $\Gamma$-hemirings

**Definition 3.1.** Let $\mu$ be a fuzzy subset of $S$ and $x \in S$. Then the fuzzy subset $< x, \mu >$ of S, defined by $< x, \mu > (y) = \inf\limits_{\gamma\in\Gamma} \mu(x\gamma y)$ for all $y \in S$, is called the *extension* of $\mu$ by x.

**Theorem 3.2.** *Let $\mu$ is a fuzzy right h-ideal of S and $x \in S$. Then the extension $< x, \mu >$ is a fuzzy right h-ideal of S.*

*Proof.* Let $p, q, a, b, z \in S$ and $\beta \in \Gamma$. Then

$$\begin{aligned} < x, \mu > (p+q) &= \inf_{\gamma\in\Gamma} \mu(x\gamma(p+q)) \\ &= \inf_{\gamma\in\Gamma} \mu(x\gamma p + x\gamma q)) \\ &\geq \inf_{\gamma\in\Gamma} \min\{\mu(x\gamma p), \mu(x\gamma q)\} \\ &= \min\{\inf_{\gamma\in\Gamma} \mu(x\gamma p), \inf_{\gamma\in\Gamma} \mu(x\gamma p)\} \\ &= \min\{< x, \mu > (p), < x, \mu > (q)\} \end{aligned}$$

Also,

$$< x, \mu > (p\beta q) = \inf_{\gamma\in\Gamma} \mu(x\gamma p\beta q) \geq \inf_{\gamma\in\Gamma} \mu(x\gamma p).$$

Now let $p + a + z = b + z$. So, $x\gamma p + x\gamma a + x\gamma z = x\gamma b + x\gamma z$. Then

$$\begin{aligned} < x, \mu > (p) &= \inf_{\gamma\in\Gamma} \mu(x\gamma p) \\ &\geq \inf_{\gamma\in\Gamma} \min\{\mu(x\gamma a), \mu(x\gamma b)\} \\ &= \min\{\inf_{\gamma\in\Gamma} \mu(x\gamma a), \inf_{\gamma\in\Gamma} \mu(x\gamma b)\} \\ &= \min\{< x, \mu > (a), < x, \mu > (b)\} \end{aligned}$$

Hence $< x, \mu >$ is a fuzzy right $h$-ideal of $S$. □

**Note.** If $\mu$ is a fuzzy $h$-ideal of a commutative $\Gamma$-hemiring $S$ and $x \in S$, then the extension $< x, \mu >$ is a fuzzy $h$-ideal of $S$.

**Proposition 3.3.** *If $\mu_i, i = 1, 2, ...$ be an arbitrary collection of fuzzy h-ideal of $S$, then $< x, \bigcap_i \mu_i >$ is also a fuzzy h-ideal of $S$.*

**Definition 3.4** ([1])**.** Let $R$ and $S$ be $\Gamma$-hemirings and $f : R \to S$ be a function. Then $f$ is said to be a $\Gamma$-*homomorphism* if

(i) $f(a + b) = f(a) + f(b)$,
(ii) $f(a\alpha b) = f(a)\alpha f(b)$ for $a, b \in R$ and $\alpha \in \Gamma$,
(iii) $f(0_R) = 0_S$ where $0_R$ and $0_S$ are the zeroes of $R$ and $S$ respectively.

**Definition 3.5** ([9])**.** Let $f$ be a function from a set $X$ to a set $Y$ and $\mu$ be a fuzzy subset of $X$ and $\sigma$ be a fuzzy subset of $Y$. Then *image* of $\mu$ under $f$, denoted by $f(\mu)$, is a fuzzy subset of $Y$ defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 \quad \text{otherwise} \end{cases}$$

The *pre-image* of $\sigma$ under $f$, symbolized by $f^{-1}(\sigma)$, is a fuzzy subset of X defined by $f^{-1}(\sigma)(x) = \sigma(f(x))$ for all $x \in X$.

**Proposition 3.6.** *Let $f : R \to S$ be a morphism of $\Gamma$-hemirings.*

(i) *If $\phi$ is a fuzzy right h-ideal of $S$, then $< z, f^{-1}(\phi) >$ is a fuzzy right h-ideal of $R$, for any $z \in R$.*
(ii) *If $f$ is surjective morphism and $\mu$ is a fuzzy right h-ideal of $R$, then $< z, f(\mu) >$ is a fuzzy right h-ideal of $S$, for any $z \in S$.*

*Proof.* Let $f : R \to S$ be a morphism of $\Gamma$-hemirings.

(i) Let $\phi$ be a fuzzy right $h$-ideal of $S$. Then by Proposition 17 of [10], we have $f^{-1}(\phi)$ is a fuzzy right $h$-ideal of $R$. Now $< z, f^{-1}(\phi) >$ is an extension of $f^{-1}(\phi)$ in $R$. So, applying Theorem 3.2 we obtain that $< z, f^{-1}(\phi) >$ is a fuzzy right $h$-ideal of $R$.

(ii) Since $f$ is surjective morphism and $\mu$ is a fuzzy right h-ideal of $R$, by Proposition 17 of [10], we have $f(\mu)$ is a fuzzy right $h$-ideal of $S$. Hence with the help of Theorem 3.2 we get that $< z, f(\mu) >$ is a fuzzy right $h$-ideal of $S$ for any $z \in S$. □

**Proposition 3.7.** *Let $\mu$ be a fuzzy h-ideal of $S$ and $x \in S$. Then the following conditions hold*

(i) $\mu \subseteq < x, \mu >$,
(ii) $< (x\gamma)^{n-1}x, \mu > \subseteq < (x\gamma)^n x, \mu >$ *where* $\gamma \in \Gamma$,
(iii) *If $\mu(x) > 0$ then supp $< x, \mu > = S$ where supp $\mu$ is defined by*

$$supp\ \mu = \{s \in S : \mu(s) > 0\}.$$

*Proof.* (i) Let $y \in S$. Now $< x, \mu > (y) = \inf\limits_{\gamma \in \Gamma} \mu(x\gamma y) \geq \mu(y)$. Thus $\mu \subseteq < x, \mu >$.

(ii) Let $n$ be a positive integer and $y \in S$. Then

$$\begin{aligned} < (x\gamma)^n x, \mu > (y) &= \inf_{\gamma\in\Gamma} \mu((x\gamma)^n x\gamma y) \\ &\geq \inf_{\gamma\in\Gamma} \mu((x\gamma)(x\gamma)^{n-1} x\gamma y) \\ &\geq \inf_{\gamma\in\Gamma} \mu((x\gamma)^{n-1} x\gamma y) \\ &= < (x\gamma)^{n-1} x, \mu > (y). \end{aligned}$$

So, $< (x\gamma)^{n-1}x, \mu > \subseteq < (x\gamma)^n x, \mu >$.

(iii) Let $\mu(x) > 0$ and $y \in S$. Then $< x, \mu > (y) = \inf_{\gamma\in\Gamma} \mu(x\gamma y) \geq \mu(x)$. Thus $y \in$ *supp* $< x, \mu >$ and consequently, $S \subseteq$ *supp* $< x, \mu >$. Hence $S =$ *supp* $< x, \mu >$. □

**Proposition 3.8.** *If $\mu$ is a fuzzy $h$-bi-ideal of $S$ then its extension by $x \in S$, $< x, \mu >$ is also a fuzzy $h$-bi-ideal of $S$ provided $S$ is commutative.*

*Proof.* Let $\mu$ be a fuzzy $h$-bi-ideal of $S$ and its extension by $x \in S$ is $< x, \mu >$. Since $\mu$ be a fuzzy $h$-bi-ideal it is sufficient to prove $< x, \mu > (p\alpha q\beta r) \geq \min\{< x, \mu > (p), < x, \mu > (r)\}$ for all $p, q, r \in S$ and $\alpha, \beta \in \Gamma$. Suppose $p, q, r \in S$ and $\alpha, \beta \in \Gamma$. Now

$$< x, \mu > (p\alpha q\beta r) = \inf_{\gamma\in\Gamma} \mu(x\gamma p\alpha q\beta r) \geq \inf_{\gamma\in\Gamma} \mu(x\gamma p) =< x, \mu > (p).$$

Also, $< x, \mu > (p\alpha q\beta r) = \inf_{\gamma\in\Gamma} \mu(x\gamma p\alpha q\beta r) \geq \inf_{\gamma\in\Gamma} \mu(x\gamma r) =< x, \mu > (r)$ (since $S$ is commutative). Therefore $< x, \mu > (p\alpha q\beta r) \geq \min\{< x, \mu > (p), < x, \mu > (r)\}$. So, $< x, \mu >$ is a fuzzy h-bi-ideal of $S$. □

**Proposition 3.9.** *If $\mu$ is a fuzzy $h$-interior-ideal of $S$ then its extension by $x \in S$, $< x, \mu >$ is also a fuzzy $h$-interior-ideal of $S$ provided $S$ is commutative and $\Gamma$ is also a $S$-hemiring.*

*Proof.* Let $\mu$ be a fuzzy $h$-interior-ideal of $S$ and its extension by $x \in S$ is $< x, \mu >$. Then it is sufficient to prove $< x, \mu > (p\alpha q\beta r) \geq < x, \mu > (q)$ for all $p, q, r \in S$ and $\alpha, \beta \in \Gamma$. Suppose $p, q, r \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} < x, \mu > (p\alpha q\beta r) &= \inf_{\gamma\in\Gamma} \mu(x\gamma p\alpha q\beta r) \\ &= \inf_{\gamma\in\Gamma} \mu(x\gamma p\alpha r\beta q) && \text{(since } S \text{ is commutative)} \\ &= \inf_{\gamma'\in\Gamma} \mu(x\gamma' q) && \text{(since } \Gamma \text{ is also a } S\text{-hemiring)} \\ &=< x, \mu > (q). \end{aligned}$$

Hence $< x, \mu >$ is a fuzzy $h$-interior-ideal of $S$. □

**Proposition 3.10.** *If $\mu$ is a fuzzy $h$-quasi-ideal of $S$ then its extension by $x \in S$, $< x, \mu >$ is also a fuzzy $h$-quasi-ideal of $S$.*

*Proof.* Let $\mu$ be a fuzzy $h$-quasi ideal of $S$ and its extension by $x \in S$ is $< x, \mu >$. Let $p, a, b, z \in S$. Then

$$< x, (\mu o_h \chi_S) \cap (\chi_S o_h \mu) > (p) = \inf_{\gamma \in \Gamma} ((\mu o_h \chi_S) \cap (\chi_S o_h \mu))(x\gamma p)$$
$$= \inf_{\gamma \in \Gamma} \min\{(\mu o_h \chi_S)(x\gamma p), (\chi_S o_h \mu)(x\gamma p)\}$$
$$\leq \inf_{\gamma \in \Gamma} \min\{\mu(x\gamma p), \mu(x\gamma p)\} \qquad \text{(since } \mu \text{ is a fuzzy } h\text{-quasi-ideal)}$$
$$= \inf_{\gamma \in \Gamma} \mu(x\gamma p) = < x, \mu > (p).$$

Also from Theorem 3.2 we have $< x, \mu > (p + q) \geq \min\{\mu(p), \mu(q)\}$ and $p + a + z = q + z$ implies $< x, \mu > (p) \geq \min\{< x, \mu > (a), < x, \mu > (b)\}$. Hence $< x, \mu >$ is a fuzzy $h$-quasi ideal of $S$. □

**Remark 3.11.** We know that if $\mu$ is fuzzy $h$-quasi-ideal of a $\Gamma$-hemiring $S$ it is also a fuzzy $h$-bi-ideal. In previous proposition 3.10 we show that its extension by any element $x \in S$, $< x, \mu >$ is a fuzzy $h$-quasi-ideal also. Now it is a routine verification to show that $< x, \mu >$ is also a fuzzy $h$-bi-ideal of $S$ provided $S$ is commutative.

**Proposition 3.12.** *Let $\mu$ be a fuzzy $h$-ideal of $S$. Then for any $x \in S$, $< x, \mu_+ >$ is also a fuzzy $h$-ideal of $S$, where $\mu_+$ is defined by $\mu_+(x) = \mu(x) - \mu(0) + 1$.*

*Proof.* Since $\mu$ is a fuzzy $h$-ideal of S, by Proposition 25 of [10] we have $\mu_+$ is also a fuzzy $h$-ideal and hence by using Theorem 3.2 we deduce that $< x, \mu_+ >$ is also a fuzzy $h$-ideal of $S$. □

**Proposition 3.13.** *If $\mu$ is a fuzzy $h$-ideal of $S$, then for any $x \in S$, $< x, \mu_{\beta,\alpha} >$ is also a fuzzy $h$-ideal of $S$, where $\mu_{\beta,\alpha}(y) = \beta\mu(y) + \alpha$, $\beta \in (0, 1]$ and $\alpha \in [0, 1 - \sup\{\mu(y) : y \in S\}]$.*

*Proof.* Since $\mu$ is a fuzzy $h$-ideal of S, by Theorem 20 of [10] we have $\mu_{\beta,\alpha}$ is also a fuzzy $h$-ideal and hence by using Theorem 3.2 we deduce that $< x, \mu_{\beta,\alpha} >$ is also a fuzzy $h$-ideal of $S$. □

**Proposition 3.14.** *If $\mu$ and $\nu$ are any two fuzzy $h$-ideal of $S$, then for any $x \in S$, $< x, \mu \times \nu >$ is also a fuzzy $h$-ideal of $S$, where $(\mu \times \nu)(a, b) = \min\{\mu(a), \mu(b)\}$, $a, b \in S$.*

*Proof.* Since $\mu$ and $\nu$ be any two fuzzy $h$-ideal of S, by Theorem 35 of [10] we have $\mu \times \nu$ is also a fuzzy $h$-ideal and hence by using Theorem 3.2 we deduce that $< x, \mu \times \nu >$ is also a fuzzy $h$-ideal of $S$. □

**Theorem 3.15.** *Let $\mu$, $\nu$ be any two fuzzy $h$-ideal of $S$ and $x, y \in S$. Then $< x, \mu > \times < y, \nu >$ is also a fuzzy $h$-ideal of $S$.*

*Proof.* Since$\mu$ and $\nu$ be any two fuzzy $h$-ideal of S, by Theorem 3.2 we have $< x, \mu >$ and $< y, \nu >$ are fuzzy $h$-ideals of $S$. Hence by using Theorem 35 of [10] we deduce that $< x, \mu > \times < y, \nu >$ is also a fuzzy $h$-ideal of $S$. □

**Proposition 3.16.** *Let $\mu$ be a prime fuzzy $h$-ideal of $S$. Then for all $x, y \in S$, $\inf_{\gamma \in \Gamma} \mu(x\gamma y) = \max[\mu(x), \mu(y)]$. Conversely, let $\mu$ be a fuzzy $h$-ideal of $S$ such that*

*$Im\ \mu = \{1, \alpha\}, \alpha \in [0, 1)$. If $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = \max[\mu(x), \mu(y)]$ for all $x, y \in S$ then $\mu$ is a prime fuzzy $h$-ideal of $S$.*

*Proof.* Let $\mu$ be a prime fuzzy $h$-ideal of $S$. Then by Theorem 2.9 we have, $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = 1$ or $\alpha$.
Case I. Let $\max[\mu(x), \mu(y)] = 1$. Then suppose that $\mu(x) = 1$. Consequently, $x \in \mu_0$. As $\mu_0$ is an $h$-ideal of S, $x\gamma y \in \mu_0$ for all $s \in S$ and $\gamma \in \Gamma$. Thus $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = 1 = \max[\mu(x), \mu(y)]$.
Case II. Let $\max[\mu(x), \mu(y)] = \alpha$. Then $\mu(x) = \mu(y) = \alpha$. This implies that $x, y \notin \mu_0$. Since $\mu_0$ is a prime $h$-ideal of S, so $x\Gamma y \nsubseteq \mu_0$. Thus there exists some $\gamma_1 \in \Gamma$ such that $x\gamma_1 y \notin \mu_0$, i.e., $\mu(x\gamma_1 y) \neq 1$. Therefore $\mu(x\gamma_1 y) = \alpha$. Thus $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = \alpha = \max[\mu(x), \mu(y)]$. For the converse part, let $x, y \in S$ such that $x\Gamma y \subseteq \mu_0$. Then $x\gamma y \in \mu_0$ for all $\gamma \in \Gamma$. So $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = 1 = \max[\mu(x), \mu(y)]$. This implies that either $\mu(x) = 1$ or $\mu(y) = 1$, i.e., either $x \in \mu_0$ or $y \in \mu_0$. Consequently, $\mu_0$ is a prime $h$-ideal of $S$ by Theorem 2.4. Hence by Theorem 2.9, $\mu$ is a prime fuzzy $h$-ideal of $S$. □

**Proposition 3.17.** *If $\mu$ is a prime (semiprime) fuzzy $h$-ideal of $S$ then its extension by $x \in S$, $< x, \mu >$ is also a prime (semiprime) fuzzy $h$-ideal of $S$.*

*Proof.* Let $\mu$ be a prime fuzzy $h$-ideal of $S$ and its extension by $x \in S$ is $< x, \mu >$. Let $p, q \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\inf_{\alpha\in\Gamma} < x, \mu > (p\alpha q) &= \inf_{\alpha\in\Gamma} \inf_{\gamma\in\Gamma} \mu(x\gamma p\alpha q) \\
&= \inf_{\gamma\in\Gamma} \{\max(\mu(x\gamma p), \mu(q))\} \qquad \text{(since } \mu \text{ is prime)} \\
&= \max\{\inf_{\gamma\in\Gamma} \mu(x\gamma p), \mu(q)\} \\
&= \max\{\mu(x), \mu(p), \mu(q)\} \\
&= \max\{\max\{\mu(x), \mu(p)\}, \max\{\mu(x), \mu(q)\}\} \\
&= \max\{\inf_{\gamma\in\Gamma} \mu(x\gamma p), \inf_{\gamma\in\Gamma} \mu(x\gamma q)\} \\
&= \max\{< x, \mu > (p), < x, \mu > (q)\}.
\end{aligned}$$

Similarly, we can prove the result for semiprime fuzzy $h$-ideal. □

**Proposition 3.18.** *Let $\mu$ be a prime fuzzy $h$-ideal of $S$ and $x \in S$. Then*

$$< x, \mu > (y) = \inf_{\substack{s\in S \\ \gamma,\delta\in\Gamma}} [< x\gamma s\delta x, \mu > (y)]$$

*for all $y \in S$.*

*Proof.* Let $y \in S$. Then

$$\inf_{\substack{s\in S\\ \gamma,\delta\in\Gamma}} [< x\gamma s\delta x, \mu > (y)] = \inf_{\substack{s\in S\\ \gamma,\delta\in\Gamma}} \inf_{\gamma_1\in\Gamma} \mu(x\gamma s\delta x\gamma_1 y)$$
$$= \inf_{\substack{s\in S\\ \gamma,\delta\in\Gamma}} \max[\mu(x\gamma s\delta x), \mu(y)] = \max\left[\inf_{\substack{s\in S\\ \gamma,\delta\in\Gamma}} \mu(x\gamma s\delta x), \mu(y)\right]$$
$$= \max[\max[\mu(x), \mu(x)], \mu(y)] = \max[\mu(x), \mu(y)]$$
$$= \inf_{\gamma\in\Gamma} \mu(x\gamma y) =< x, \mu > (y).$$

This completes the proof. □

**Definition 3.19.** Let $A \subseteq S$ and $x \in S$. Then we define $< x, A >$ as

$$< x, A >= \{y \in S : x\Gamma y \subseteq A\}.$$

**Proposition 3.20.** *Let $A$ be a non empty subset of $S$. Then $< x, \lambda_A >= \lambda_{<x,A>}$ for every $x \in S$ where $\lambda_A$ denotes the characteristic function of $A$.*

*Proof.* Let $y \in S$. Then $< x, \lambda_A > (y) = \inf_{\gamma\in\Gamma} \lambda_A(x\gamma y) = 1$ or $0$. If $< x, \lambda_A > (y) = 1$ then $\lambda_A(x\gamma y) = 1$ for all $\gamma \in \Gamma$. Thus $x\gamma y \in A$ for all $\gamma \in \Gamma$, i.e., $x\Gamma y \subseteq A$. i.e., $y \in< x, A >$. Consequently $\lambda_{<x,A>}(y) = 1$. So, $< x, \lambda_A >= \lambda_{<x,A>}$. If $< x, \lambda_A > (y) = 0$ then $\lambda_A(x\gamma y) = 0$ for some $\gamma \in \Gamma$. So $x\gamma y \notin A$ for some $\gamma \in \Gamma$. Therefore $x\Gamma y \nsubseteq A$ which implies that $y \notin< x, A >$. Thus $\lambda_{<x,A>}(y) = 0$. Hence $< x, \lambda_A >= \lambda_{<x,A>}$. This proves the proposition. □

**Theorem 3.21.** *Let $\mu$ be a prime fuzzy $h$-ideal of $S$ and $x \in S$ be such that $x \notin \mu_0$. Then $< x, \mu >= \mu$. Conversely, let $\mu$ be a fuzzy $h$-ideal of $S$ such that $Im\ \mu = \{1, \alpha\}$, $\alpha \in [0, 1)$. If $< x, \mu >= \mu$ for those $x \in S$ for which $\mu(x) = \alpha$, then $\mu$ is a prime fuzzy $h$-ideal of $S$.*

*Proof.* Let $\mu$ be a fuzzy prime $h$-ideal of $S$. Then by Theorem 2.9,

(i) $\mu(0) = 1$,
(ii) $Im\ \mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$,
(iii) $\mu_0 = \{x \in S : \mu(x) = \mu(0)\}$ is a prime $h$-ideal of $S$.

Case I. If $y \in \mu_0$ then $x\gamma y \in \mu_0$ for all $\gamma \in \Gamma$. So $\inf_{\gamma\in\Gamma} \mu(x\gamma y) = 1 = \mu(y)$. Therefore $< x, \mu >= \mu$.

Case II. Let $y \notin \mu_0$. Since $\mu_0$ is a prime $h$-ideal of $S$ and $x, y \notin \mu_0$, so $x\Gamma y \nsubseteq \mu_0$. Therefore there exists some $\gamma \in \Gamma$ such that $x\gamma y \notin \mu_0$. Therefore $\mu(x\gamma y) = \alpha$. Thus

$$\inf_{\gamma\in\Gamma} \mu(x\gamma y) = \alpha = \mu(y)$$

and hence $< x, \mu >= \mu$. Consequently, $< x, \mu >= \mu$ for all $x \in \mu_0$.

Conversely, let $x, y \in S$. For the case $\mu(x) = \alpha$, we have $\max\{\mu(x), \mu(y)\} = \mu(y)$. Now $\mu(y) =< x, \mu > (y)$ implies that $\max\{\mu(x), \mu(y)\} = \inf_{\gamma\in\Gamma} \mu(x\gamma y)$. The case $\mu(x) = 1$ implies that $x \in \mu_0$. So $x\gamma y \in \mu_0$ for all $\gamma \in \Gamma$. Thus

$$\inf_{\gamma\in\Gamma} \mu(x\gamma y) = 1 = \mu(x) = \max\{\mu(x), \mu(y)\}.$$

Thus $\inf_{\gamma\in\Gamma}\mu(x\gamma y) = \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$. Hence by the converse part of Theorem 3.16, $\mu$ is a prime $h$-ideal of $S$. □

**Theorem 3.22.** *Let $\mu$ be a prime fuzzy $h$-ideal of $S$ and $x \in S$ be such that $x \in \mu_0$. Then $< x, \mu >= \mathbf{1}_S$.*

*Proof.* Let $\mu$ be a prime fuzzy $h$-ideal of $S$ and $y \in S$. Then $x\gamma y \in \mu_0$ for all $s \in S$ and for all $\gamma \in \Gamma$ as $x \in \mu_0$. So $< x, \mu > (y) = \inf_{\gamma\in\Gamma}\mu(x\gamma y) = 1 = \mathbf{1}_S(y)$ for all $y \in S$. Hence $< x, \mu >= \mathbf{1}_S$. □

**Corollary 3.23.** *Let $I$ be an $h$-ideal of $S$. If $I$ is prime $h$-ideal of $S$ then for $x \in S$, $< x, \lambda_I >= \lambda_I$ where $x \notin I$.*

*Proof.* Let I be a prime $h$-ideal of $S$. Then $\lambda_I$ is a prime fuzzy $h$-ideal of $S$. Now $x \notin I$ implies that $x \notin (\lambda_I)_0$. Hence by Theorem 3.21, $< x, \lambda_I >= \lambda_I$. □

**Theorem 3.24.** *Let $S$ be a commutative $\Gamma$-hemiring and $\mu$ be a fuzzy subset of $S$ such that $< x, \mu >= \mu$ for all $x \in S$. Then $\mu$ is constant.*

*Proof.* Let $x, y \in S$. Then

$$\begin{aligned}\mu(y) &=< x, \mu > (y) = \inf_{\gamma\in\Gamma}\mu(x\gamma y)\\ &= \inf_{\gamma\in\Gamma}\mu(y\gamma x) \qquad \text{(since } S \text{ is a commutative } \Gamma\text{-hemiring)}\\ &=< y, \mu > (x) = \mu(x).\end{aligned}$$

Therefore $\mu(x) = \mu(y)$ for all $x, y \in S$. Hence $\mu$ is constant. □

## 4. Conclusions

Throughout this paper in some cases, we consider $\Gamma$-hemiring S to be both-sided. A careful reader may arise confusion to the definition of extension, in $\Gamma$-hemiring with $\Gamma$-semigroup and may feel some difficulty when he/she read the paper taking $\Gamma$-hemiring $S$ as one-sided. We can solve this problem if we replace Definition 3.1 by $< x, \mu > (y) = \inf_{\substack{s\in S\\ \alpha,\gamma\in\Gamma}} \mu(x\alpha s\gamma y)$. Then most of the above said results hold if we change the proofs accordingly. This new definition will be more appropriate and be fruitful in studying $\Gamma$-hemiring via its operator hemirings. In our next paper, we follow the above definition.

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